Geometry Refresher

MATHCOUNTS

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Areas of plane figures

Areas of plane figures: square, rectangle, rhombus, parallelogram, trapezoid, quadrangle, right-angled triangle, isosceles triangle, equilateral triangle, arbitrary triangle, polygon, regular hexagon, circle, sector, segment of a circle. Heron's formula.

A square (Fig. 58). \(a\) – a side, \(d\) – a diagonal.

\[ S = a^2 = d^2 / 2 . \]

A rectangle (Fig. 59). \(a, b\) – sides.

\[ S = ab . \]

A rhombus (Fig. 60) \(a\) – a side; \(b, c\) – diagonals; \(\alpha\) – one of angles.

\[ S = bc / 2 = a^2 \sin \alpha . \]

A parallelogram (Fig. 61) \(a, b\) – sides; \(\alpha\) – one of angles, \(h\) – a height

\[ S = ah = ab \sin \alpha . \]

A trapezoid (Fig. 62) \(a, b\) – bases; \(h\) – a height

\[ S = \frac{a + b}{2} \cdot h . \]
Any quadrangle (Fig.63). \( a, b \) – diagonals; \( \alpha \) – an angle between them.

\[
S = \frac{1}{2} ab \sin \alpha .
\]

A quadrangle, around of which a circle can be circumscribed. \( a, b, c, d \) – sides.

\[
S = \sqrt{(p-a)(p-b)(p-c)(p-d)} , \quad p = \frac{a + b + c + d}{2} .
\]

A right angled triangle (Fig.64).

\[
S = \frac{ab}{2} .
\]

An isosceles triangle (Fig.65). \( a \) – a base; \( b \) – a lateral side.

\[
S = \frac{a}{2} \sqrt{b^2 - a^2/4} .
\]

An equilateral triangle (Fig.66). \( a \) – a side.

\[
S = \frac{\sqrt{3}}{4} a^2 .
\]

Any triangle. \( a, b, c \) – sides; \( a \) – a base; \( h \) – a height; \( A, B, C \) – angles, opposite to sides \( a, b, c \); \( p = (a + b + c)/2 \).
The last expression is known as Heron's formula.

A polygon, area of which we want to determine, can be divided into some triangles by its diagonals. A polygon, circumscribed around a circle (Fig. 67), can be divided by lines, going from a center of a circle to its vertices. Then we receive:

\[
S = \frac{1}{2} \cdot a \cdot h = \frac{1}{2} \cdot ab \cdot \sin C = \frac{a^2 \cdot \sin B \cdot \sin C}{2 \cdot 2 \sin A} = \frac{h^2 \cdot \sin A}{2 \cdot 2 \sin B \cdot \sin C} = \sqrt{p(p-a)(p-b)(p-c)}.
\]

Particularly, this formula is valid for any regular polygon.

A regular hexagon. \(a\) – a side.

\[
S = \frac{3}{2} \sqrt{3} a^2.
\]

A circle. \(D\) – a diameter; \(r\) – a radius.

\[
S = \pi r^2 = \pi D^2 / 4.
\]

A sector (Fig. 68). \(r\) – a radius; \(n\) – a degree measure of a central angle; \(l\) – a length of an arc.
A segment (Fig.68). An area of a segment is found as a difference between areas of a sector \( \text{AmBO} \) and a triangle \( \text{AOB} \). Besides, the approximate formula for an area of a segment is:

\[
S \approx \frac{2}{3}ah,
\]

where \( a = AB \) (Fig.68) – a base of segment; \( h \) – its height (\( h = r - OD \)). A relative error of this formula is equal: at \( \text{AmB} = 60 \, \text{deg} \) – about 1.5% ; at \( \text{AmB} = 30 \, \text{deg} \) ~0.3%.

Example. Calculate areas of the sector \( \text{AmBO} \) (Fig.68) and the segment \( \text{AmB} \) at the following data: \( r = 10 \, \text{cm} \), \( n = 60 \, \text{deg} \).

Solution. A sector area:

\[
S_1 = \frac{\pi r^2 n}{360} = \frac{\pi \cdot 10^2 \cdot 60}{360} \approx 52.36 \, \text{cm}^2.
\]

An area of the regular triangle \( \text{AOB} \):

\[
S_2 = \frac{\sqrt{3}}{4} a^2 = \frac{\sqrt{3}}{4} 10^2 \approx 43.30 \, \text{cm}^2.
\]

Hence, an area of a segment:

\[
S = S_1 - S_2 = 52.36 - 43.30 = 9.06 \, \text{cm}^2.
\]

Note, that in a regular triangle \( \text{AOB} \): \( AB = AO = BO = r \), \( AD = BD = r / 2 \), and therefore a height \( OD \) according to
Pythagorean theorem is equal to:

$$OD = \sqrt{AO^2 - AD^2} = \sqrt{r^2 - r^2/4} = r \frac{\sqrt{3}/4}{2} = r \frac{\sqrt{3}}{2}.$$ 

Then, according to the approximate formula we’ll receive:

$$S = \frac{2}{3} AB (r - OD) = -\frac{2}{3} (r - r \frac{\sqrt{3}}{2}) \approx \frac{2}{3} \frac{10^2 (1 - 0.866)}{} = 8.93 \text{ cm}^2.$$ 

A relative error $\varepsilon$ is equal to:

$$\varepsilon = \frac{9.06 - 8.93}{9.06} \times 100 \approx 1.5\%.$$ 

**Angles**

*Angle. Degree and radian measures of an angle.*

*Right (direct), acute and obtuse angle. Mutually perpendicular straight lines. Signs of angles.*

*Supplementary (adjacent) angles. Vertically opposite (vertical) angles. Bisector of an angle.*

*Angle* is a geometric figure (Fig. 1), formed by two rays $OA$ and $OB$ (sides of an angle), going out of the same point $O$ (a vertex of an angle).

An angle is signed by the symbol $\angle$ and three letters, marking ends of rays and a vertex of an angle: $\angle AOB$ (moreover, a vertex letter is placed in the middle). A measure of an angle is a value of a turn around a vertex $O$, that transfers a ray $OA$ to the position $OB$. Two units of angles measures are widely used: a *radian* and a *degree*. About a radian measure see below in the point “A length of arc” and also in the section “Trigonometry”.

A *degree measure*. Here a unit of measurement is a *degree* (its designation is $^\circ$ or $\deg$) – a turn
of a ray by the 1/360 part of the one complete revolution. So, the complete revolution of a ray is equal to 360 deg. One degree is divided by 60 minutes (a designation is ‘ or min); one minute – correspondingly by 60 seconds (a designation is “ or sec). An angle of 90 deg (Fig.2) is called a right or direct angle; an angle lesser than 90 deg (Fig.3), is called an acute angle; an angle greater than 90 deg (Fig.4), is called an obtuse angle.

![Fig. 2](image1.png) ![Fig. 3](image2.png) ![Fig. 4](image3.png)

Straight lines, forming a right angle, are called mutually perpendicular lines. If the straight lines AB and MK are perpendicular, this is signed as: AB \perp MK.

**Signs of angles.** An angle is considered as positive, if a rotation is executed opposite a clockwise, and negative – otherwise. For example, if the ray OA displaces to the ray OB as shown on Fig.2, then ∠AOB = + 90 deg; but on Fig.5 ∠AOB = − 90 deg.

![Fig. 5](image4.png) ![Fig. 6](image5.png)

**Supplementary (adjacent) angles** (Fig.6) – angles AOB and COB, having the common vertex O and the common side OB; other two sides OA and OC form a continuation one to another. So, a sum of supplementary (adjacent) angles is equal to 180 deg.

**Vertically opposite (vertical) angles** (Fig.7) – such two angles with a common vertex, that sides of one angle are continuations of the other: ∠AOB and ∠COD (and also ∠AOC and ∠DOB) are vertical angles.

![Fig. 7](image6.png) ![Fig. 8](image7.png)

A bisector of an angle is a ray, dividing the angle in two (Fig.8). Bisectors of vertical angles (OM and ON, Fig.9) are continuations one of the other. Bisectors of supplementary angles (OM and ON, Fig.10) are mutually perpendicular lines.
The property of an angle bisector: any point of an angle bisector is placed by the same distance from the angle sides.

Parallel straight lines


Two straight lines AB and CD (Fig.11) are called parallel straight lines, if they lie in the same plane and don’t intersect however long they may be continued. The designation: AB||CD. All points of one line are equidistant from another line. All straight lines, parallel to one straight line are parallel between themselves. It’s adopted that an angle between parallel straight lines is equal to zero. An angle between two parallel rays is equal to zero, if their directions are the same, and 180 deg, if the directions are opposite. All perpendiculars (AB, CD, EF, Fig.12) to the one straight line KM are parallel between themselves. Inversely, the straight line KM, which is perpendicular to one of parallel straight lines, is perpendicular to all others. A length of perpendicular segment, concluded between two parallel straight lines, is a distance between them.

At intersecting two parallel straight lines by the third line, eight angles are formed (Fig.13), which are called two-by-two:
1) *corresponding angles* (1 and 5; 2 and 6; 3 and 7; 4 and 8); these angles are equal two-by-two: ($\angle 1 = \angle 5; \angle 2 = \angle 6; \angle 3 = \angle 7; \angle 4 = \angle 8$);

2) *alternate interior angles* (4 and 5; 3 and 6); they are equal two-by-two;

3) *alternate exterior angles* (1 and 8; 2 and 7); they are equal two-by-two;

4) *one-sided interior angles* (3 and 5; 4 and 6); a sum of them two-by-two is equal to 180 deg ($\angle 3 + \angle 5 = 180$ deg; $\angle 4 + \angle 6 = 180$ deg);

5) *one-sided exterior angles* (1 and 7; 2 and 8); a sum of them two-by-two is equal to 180 deg ($\angle 1 + \angle 7 = 180$ deg; $\angle 2 + \angle 8 = 180$ deg).

Angles with *correspondingly parallel sides* either are equal one to another, (if both of them are acute or both are obtuse, $\angle 1 = \angle 2$, Fig.14), or sum of them is 180 deg ($\angle 3 + \angle 4 = 180$ deg, Fig.15).

Angles with *correspondingly perpendicular sides* are also either equal one to another (if both of them are acute or both are obtuse), or sum of them is 180 deg.
**Thales' theorem.** At intersecting sides of an angle by parallel lines (Fig.16), the angle sides are divided into the proportional segments:

\[
\begin{align*}
\frac{OA}{OA'} &= \frac{OC}{OC'} &= \frac{AB}{A'B'} &= \frac{BC}{B'C'} &= \frac{AC}{A'C'} \quad .
\end{align*}
\]

**Euclidean geometry axioms**

*Axiom of belonging.* Axiom of ordering.

*Axiom of congruence (equality) of segments and angles.*

*Axiom of parallel straight lines.*

*Archimedean axiom of continuity.*

As we have noted above, there is a set of the axioms – properties, that are considered in geometry as main ones and are adopted without a proof. Now, after introducing some initial notions and definitions we can consider the following sufficient set of the axioms, usually used in plane geometry.

**Axiom of belonging.** Through any two points in a plane it is possible to draw a straight line, and besides only one.

**Axiom of ordering.** Among any three points placed in a straight line, there is no more than one point placed between the two others.

**Axiom of congruence (equality) of segments and angles.** If two segments (angles) are congruent to the third one, then they are congruent to each other.

**Axiom of parallel straight lines.** Through any point placed outside of a straight line it is possible to draw another straight line, parallel to the given line, and besides only one.

**Axiom of continuity (Archimedean axiom).** Let AB and CD be two some segments; then there is a finite set of such points \( A_1, A_2, \ldots, A_n \), placed in the straight line AB, that segments \( AA_1, A_1A_2, \ldots, A_{n-1}A_n \) are congruent to segment CD, and point B is placed between A and \( A_n \).

We emphasize, that replacing one of these axioms by another, turns this axiom into a theorem, requiring a proof. So, instead of the axiom of parallel straight lines we can use as an axiom the property of triangle angles (“the sum of triangle angles is equal to 180 deg”). But then we should to prove the property of parallel lines.

**Polygon**

*Polygon. Vertices, angles, diagonals, sides of a polygon.*

*Perimeter of a polygon. Simple polygons. Convex polygon.*

*Sum of interior angles in a convex polygon.*

A plane figure, formed by closed chain of segments, is called a **polygon**. Depending on a quantity of angles a polygon can be a **triangle**, a **quadrangle**, a **pentagon**, a **hexagon** etc. On Fig.17 the hexagon ABCDEF is shown. Points
A, B, C, D, E, F – vertices of polygon; angles $\angle A$, $\angle B$, $\angle C$, $\angle D$, $\angle E$, $\angle F$ – angles of polygon; segments AC, AD, BE etc. are diagonals; AB, BC, CD, DE, EF, FA – sides of polygon; a sum of sides lengths AB + BC + … + FA is called a perimeter of polygon and signed as $p$ (sometimes $2p$, then $p$ – a half-perimeter). We consider only simple polygons in an elementary geometry, contours of which have no self-intersections (as shown on Fig.18). If all diagonals lie inside of a polygon, it is called a convex polygon. A hexagon on Fig.17 is a convex one; a pentagon ABCDE on Fig.19 is not a convex polygon, because its diagonal AD lies outside of it. A sum of interior angles in any convex polygon is equal to $180(n-2)$ deg, where $n$ is a number of angles (or sides) of a polygon.

**Triangle**


Scalene triangle. Main properties of triangles. Theorems about congruence of triangles. Remarkable lines and points of a triangle.

Pythagorean theorem. Relation of sides for an arbitrary triangle.

**Triangle** is a polygon with three sides (or three angles). Sides of triangle are signed often by small letters, corresponding to designations of opposite vertices, signed by capital letters.

If all the three angles are acute (Fig.20), then this triangle is an acute-angled triangle; if one of the angles is right ($\angle C$, Fig.21), then this triangle is a right-angled triangle; sides a, b, forming a right angle, are called legs; side c, opposite to a right angle, called a hypotenuse; if one of the angles is obtuse ($\angle B$, Fig.22), then this triangle is an obtuse-angled triangle.
A triangle ABC is an isosceles triangle (Fig.23), if the two of its sides are equal \( a = c \); these equal sides are called lateral sides, the third side is called a base of triangle. A triangle ABC is an equilateral triangle (Fig.24), if all of its sides are equal \( a = b = c \). In general case \( a \ b \ c \) we have a scalene triangle.

**Main properties of triangles.** In any triangle:

1. An angle, lying opposite the greatest side, is also the greatest angle, and inversely.

2. Angles, lying opposite the equal sides, are also equal, and inversely. In particular, all angles in an equilateral triangle are also equal.

3. A sum of triangle angles is equal to 180 deg.

   From the two last properties it follows, that each angle in an equilateral triangle is equal to 60 deg.

4. Continuing one of the triangle sides (AC, Fig. 25), we receive an exterior angle \( \angle BCD \).

   An exterior angle of a triangle is equal to a sum of interior angles, not supplementary with it: \( \angle BCD = \angle A + \angle B \).

5. Any side of a triangle is less than a sum of two other sides and more than their difference \( a < b + c, \ b < a + c, \ c < b + a \) (\( a < b + c, \ b > a - c, \ c < a + b, \ c > a - b \)).

**Theorems about congruence of triangles.**

Two triangles are congruent, if they have accordingly equal:

a) two sides and an angle between them;

b) two angles and a side, adjacent to them;

c) three sides.

**Theorems about congruence of right-angled triangles.**

Two right-angled triangles are congruent, if one of the following conditions is valid:
1) their legs are equal;

2) a leg and a hypotenuse of one of triangles are equal to a leg and a hypotenuse of another;

3) a hypotenuse and an acute angle of one of triangles are equal to a hypotenuse and an acute angle of another;

4) a leg and an adjacent acute angle of one of triangles are equal to a leg and an adjacent acute angle of another;

5) a leg and an opposite acute angle of one of triangles are equal to a leg and an opposite acute angle of another.

**Remarkable lines and points of triangle.**

_Altitude (height)_ of a triangle is a perpendicular, dropped from any vertex to an opposite side (or to its continuation). This side is called a base of triangle in this case. Three heights of triangle always intersect in one point, called an _orthocenter_ of a triangle. An orthocenter of an acute-angled triangle (point O, Fig.26) is placed inside of the triangle; and an orthocenter of an obtuse-angled triangle (point O, Fig.27) – outside of the triangle; an orthocenter of a right-angled triangle coincides with a vertex of the right angle.

中途線 is a segment, joining any vertex of triangle and a midpoint of the opposite side. Three medians of triangle (AD, BE, CF, Fig.28) intersect in one point O (always lied inside of a triangle), which is a center of gravity of this triangle. This point divides each median by ratio 2:1, considering from a vertex.

_Bisector_ is a segment of the angle bisector, from a vertex to a point of intersection with an opposite side. Three bisectors of a triangle (AD, BE, CF, Fig.29) intersect in the one point (always lied inside of triangle), which is a center of an inscribed circle (see the section “Inscribed and circumscribed polygons”).
A bisector divides an opposite side into two parts, proportional to the adjacent sides; for instance, on Fig.29 $AE : CE = AB : BC$.

**Midperpendicular** is a perpendicular, drawn from a middle point of a segment (side). Three midperpendiculars of a triangle (ABC, Fig.30), each drawn through the middle of its side (points K, M, N, Fig.30), intersect in one point O, which is a center of circle, circumscribed around the triangle (circumcircle).

![Fig. 28](image1.png)

![Fig. 29](image2.png)

In an acute-angled triangle this point lies inside of the triangle; in an obtuse-angled triangle - outside of the triangle; in a right-angled triangle - in the middle of the hypotenuse. An orthocenter, a center of gravity, a center of an inscribed circle and a center of a circumcircle coincide only in an equilateral triangle.

**Pythagorean theorem.** In a right-angled triangle a square of the hypotenuse length is equal to a sum of squares of legs lengths.

A proof of Pythagorean theorem is clear from Fig.31. Consider a right-angled triangle ABC with legs $a, b$ and a hypotenuse $c$. 
Build the square AKMB, using hypotenuse AB as its side. Then continue sides of the right-angled triangle ABC so, to receive the square CDEF, the side length of which is equal to \(a + b\). Now it is clear, that an area of the square CDEF is equal to \((a + b)^2\). On the other hand, this area is equal to a sum of areas of four right-angled triangles and a square AKMB, that is

\[c^2 + 4 \left( \frac{ab}{2} \right) = c^2 + 2ab,\]

hence,

\[c^2 + 2ab = (a + b)^2,\]

and finally, we have:

\[c^2 = a^2 + b^2.\]

**Relation of sides’ lengths for arbitrary triangle.**

In general case (for any triangle) we have:

\[c^2 = a^2 + b^2 - 2ab \cdot \cos C,\]

where \(C\) – an angle between sides \(a\) and \(b\).

**Parallelogram and trapezoid**

Parallelogram. Properties of a parallelogram.
Square. Trapezoid. Isosceles trapezoid.
Midline of a trapezoid and a triangle.

**Parallelogram** (ABCD, Fig.32) is a quadrangle, opposite sides of which are two-by-two parallel.
Any two opposite sides of a parallelogram are called *bases*, a distance between them is called a *height* (BE, Fig.32).

**Properties of a parallelogram.**

1. Opposite sides of a parallelogram are equal *(AB = CD, AD = BC).*

2. Opposite angles of a parallelogram are equal *(∠A = ∠C, ∠B = ∠D).*

3. Diagonals of a parallelogram are divided in their intersection point into two *(AO = OC, BO = OD).*

4. A sum of squares of diagonals is equal to a sum of squares of four sides:
   \[AC^2 + BD^2 = AB^2 + BC^2 + CD^2 + AD^2.\]

**Signs of a parallelogram.**

A quadrangle is a parallelogram, if one of the following conditions takes place:

1. Opposite sides are equal two-by-two *(AB = CD, AD = BC).*

2. Opposite angles are equal two-by-two *(∠A = ∠C, ∠B = ∠D).*

3. Two opposite sides are equal and parallel *(AB = CD, AB || CD).*

4. Diagonals are divided in their intersection point into two *(AO = OC, BO = OD).*

**Rectangle.**

If one of angles of parallelogram is right, then all angles are right (why?). This parallelogram is called a *rectangle* (Fig.33).
**Main properties of a rectangle.**

Sides of rectangle are its heights simultaneously.

**Diagonals of a rectangle are equal:** $AC = BD$.

*A square of a diagonal length is equal to a sum of squares of its sides’ lengths* (see above Pythagorean theorem):

$$AC^2 = AD^2 + DC^2.$$

**Rhombus.** If all sides of parallelogram are equal, then this parallelogram is called a *rhombus* (Fig.34).

Diagonals of a rhombus are mutually perpendicular ($AC \perp BD$) and divide its angles into two ($\angle DCA = \angle BCA$, $\angle ABD = \angle CBD$ etc.).

**Square** is a parallelogram with right angles and equal sides (Fig.35). *A square is a particular case of a rectangle and a rhombus simultaneously; so, it has all their above mentioned properties.*

**Trapezoid** is a quadrangle, two opposite sides of which are parallel (Fig.36).

Here $AD \parallel BC$. Parallel sides are called *bases* of a trapezoid, the two others ($AB$ and $CD$) – *lateral sides*. A distance between bases ($BM$) is a *height*. The segment $EF$, joining midpoints $E$ and $F$ of the lateral sides, is called a *midline* of a trapezoid.

*A midline of a trapezoid is equal to a half-sum of bases:*
\[
\frac{AD + BC}{EF} = \frac{1}{2}
\]

and parallel to them: \( EF \parallel AD \) and \( EF \parallel BC \).

A trapezoid with equal lateral sides (\( AB = CD \)) is called an isosceles trapezoid. In an isosceles trapezoid angles by each base, are equal (\( \angle A = \angle D, \angle B = \angle C \)). A parallelogram can be considered as a particular case of trapezoid.

**Midline of a triangle** is a segment, joining midpoints of lateral sides of a triangle. A midline of a triangle is equal to half of its base and parallel to it. This property follows from the previous part, as triangle can be considered as a limit case (“degeneration”) of a trapezoid, when one of its bases transforms to a point.

**Similarity of plane figures. Similarity criteria of triangles**

*Similarity of plane figures. Ratio of similarity.*

*Similarity of polygons. Similarity of triangles.*

*Similarity of right-angled triangles.*

**Similarity of plane figures.** If to change (to increase or to decrease) all sizes of a plane figure in the same ratio (ratio of similarity), then an old and a new figures are called similar ones. For example, a picture and its photograph are similar figures.

In two similar figures any corresponding angles are equal, that is, if points \( A, B, C, D \) of one figure correspond to points \( a, b, c, d \) of another figure, then \( \angle ABC = \angle abc, \angle BCD = \angle bcd \) and so on. **Two polygons** (\( ABCDEF \) and \( abcdef \), Fig.37) are similar, if their angles are equal: \( \angle A = \angle a, \angle B = \angle b, ..., \angle F = \angle f \), and sides are proportional:

\[
\frac{AB}{ab} = \frac{BC}{bc} = \frac{CD}{cd} = \frac{FA}{fa}.
\]

*Only proportionality of sides is not enough for similarity of polygons.* For example, the square \( ABCD \) and the rhombus \( abcd \) (Fig.38) have proportional sides: each side of the square is twice more than of the rhombus, but the diagonals have not changed proportionally.
But, for similarity of triangles proportionality of its sides is enough.

**Similarity criteria of triangles.** Two triangles are similar, if:

1) all their corresponding angles are equal;

2) all their sides are proportional;

3) two sides of one triangle are proportional to two sides of another and the angles concluded between these sides are equal.

Two right-angled triangles are similar, if

1) their legs are proportional;

2) a leg and a hypotenuse of one triangle are proportional to a leg and a hypotenuse of another;

3) two angles of one triangle are equal to two angles of another.

Areas of similar figures are proportional to squares of their resembling lines (for instance, sides). So, areas of circles are proportional to ratio of squares of diameters (or radii).

**Example.** A round metallic disc by diameter 20 cm weighs 6.4 kg. What is the weight of a round metallic disc by diameter 10 cm?

**Solution.** Because the material and the thick of a new disc are the same, the weights of the discs are proportional to their areas, and a ratio of an area of the small disc to an area of the big disc is equal to:

\[
\left( \frac{10}{20} \right)^2 = 0.25.
\]

Hence, the weight of the small disc is \(6.4 \times 0.25 = 1.6\) kg.

**Geometrical locus. Circle and circumference**


Segment of a circle. Sector of a circle. Angles in a circle.

Central angle. Inscribed angle. Circumscribed angle.

Radian measure of angles. Round angle. Ratio of
Geometrical locus (or simply locus) is a totality of all points, satisfying the certain given conditions.

Example 1. A midperpendicular of any segment is a locus, i.e. a totality of all points, equally removed from the bounds of the segment. Suppose that $PO \perp AB$ and $AO = OB$:

![Diagram](image)

Then, distances from any point $P$, lying on the midperpendicular $PO$, to bounds $A$ and $B$ of the segment $AB$ are both equal to $d$. So, each point of a midperpendicular has the following property: it is removed from the bounds of the segment at equal distances.

Example 2. An angle bisector is a locus, that is a totality of all points, equally removed from the angle sides.

Example 3. A circumference is a locus, that is a totality of all points (one of them - $A$), equally removed from its center $O$.

Circumference is a geometrical locus in a plane, that is a totality of all points, equally removed from its center. Each of the equal segments, joining the center with any point of a circumference is called a radius and signed as $r$ or $R$. A part of a plane inside of a circumference, is called a circle. A part of a circumference (for instance, $AmB$, Fig.39) is called an arc of a circle. The straight line $PQ$, going through two points $M$ and $N$ of a circumference, is called a secant (or transversal). Its segment $MN$, lying inside of the circumference, is called a chord.
A chord, going through a center of a circle (for instance, BC, Fig.39), is called a **diameter** and signed as $d$ or $D$. A diameter is the greatest chord of a circle and equal to two radii ($d = 2r$).

**Tangent.** Assume, that the secant PQ (Fig.40) is going through points K and M of a circumference. Assume also, that point M is moving along the circumference, approaching the point K. Then the secant PQ will change its position, rotating around the point K. As approaching the point M to the point K, the secant PQ tends to some limit position AB. The straight line AB is called a **tangent line** or simply a **tangent** to the circumference in the point K. The point K is called a **point of tangency**. A tangent line and a circumference have only one common point – a **point of tangency**.

**Properties of tangent.**

1) A tangent to a circumference is perpendicular to a radius, drawing to a point of tangency (AB $\perp$ OK, Fig.40).

2) From a point, lying outside a circle, it can be drawn two tangents to the same circumference; their segments lengths are equal (Fig.41).

**Segment of a circle** is a part of a circle, bounded by the arc ACB and the corresponding chord AB (Fig.42). A length of the perpendicular CD, drawn from a midpoint of the chord AB until intersecting with the arc ACB, is called a **height** of a circle segment. **Sector of a circle** is a part of a circle, bounded by the arc $AmB$ and two radii OA and OB, drawn to the ends of the arc (Fig.43).
Angles in a circle. A central angle – an angle, formed by two radii of the circle ( \(\angle AOB\), Fig.43 ). An inscribed angle – an angle, formed by two chords AB and AC, drawn from one common point ( \(\angle BAC\), Fig.44 ).

A circumscribed angle – an angle, formed by two tangents AB and AC, drawn from one common point ( \(\angle BAC\), Fig.41 ).

A length of arc of a circle is proportional to its radius \(r\) and the corresponding central angle \(\alpha\):

\[ l = \alpha r \]

So, if we know an arc length \(l\) and a radius \(r\), then the value of the corresponding central angle \(\alpha\) can be determined as their ratio:

\[ \alpha = l / r \]

This formula is a base for definition of a radian measure of angles. So, if \(l = r\), then \(\alpha = 1\), and we say, that an angle \(\alpha\) is equal to 1 radian ( it is designed as \(\alpha = 1 \text{ rad}\) ). Thus, we have the following definition of a radian measure unit: A radian is a central angle ( \(\angle AOB\), Fig.43 ), whose arc’s length is equal to its radius ( \(\text{AmB} = AO\), Fig.43 ). So, a radian measure of any angle is a ratio of a length of an arc, drawn by an arbitrary radius and concluded between the sides of this angle, to the radius of the arc. Particularly, according to the formula for a length of an arc, a length of a circumference \(C\) can be expressed as:

\[ C = 2\pi r, \]

where \(\pi\) is determined as ratio of \(C\) and a diameter of a circle \(2r\):

\[ \pi = C / 2 r. \]

\(\pi\) is an irrational number; its approximate value is 3.1415926…

On the other hand, \(2\pi\) is a round angle of a circumference, which in a degree measure is equal to
360 deg. In practice it often occurs, that both radius and angle of a circle are unknown. In this case, an arc length can be calculated by the approximate Huygens’ formula:

\[ p \approx 2l + \frac{(2l - L)}{3}, \]

where (according to Fig.42): \( p \) – a length of the arc ACB; \( l \) – a length of the chord AC; \( L \) – a length of the chord AB. If an arc contains not more than 60 deg, a relative error of this formula is less than 0.5%.

**Relations between elements of a circle.** An inscribed angle (\( \angle ABC \), Fig.45) is equal to a half of the central angle (\( \angle AOC \), Fig.45), based on the same arc AmC. Therefore, all inscribed angles (Fig.45), based on the same arc (AmC, Fig.45), are equal. As a central angle contains the same quantity of degrees, as its arc (AmC, Fig.45), then any inscribed angle is measured by a half of an arc, which is based on (AmC in our case).

![Fig. 45](image1)

All inscribed angles, based on a semi-circle (\( \angle APB, \angle AQB, \ldots \), Fig.46), are right angles (Prove this, please!). An angle (\( \angle AOD \), Fig.47), formed by two chords (AB and CD), is measured by a semi-sum of arcs, concluded between its sides: (AnD + CmB) / 2.

![Fig. 46](image2)

An angle (\( \angle AOD \), Fig.48), formed by two secants (AO and OD), is measured by a semi-difference of arcs, concluded between its sides: (AnD – BmC) / 2. An angle (\( \angle DCB \), Fig.49), formed by a tangent and a chord (AB and CD), is measured by a half of an arc, concluded
inside of it: \( \frac{\text{CmD}}{2} \). An angle (\( \angle \text{BOC} \), Fig.50), formed by a tangent and a secant (CO and BO), is measured by a semi-difference of arcs, concluded between its sides: \( \frac{\text{BmC} - \text{CnD}}{2} \).

![Fig. 49](image)

A circumscribed angle (\( \angle \text{AOC} \), Fig.50), formed by the two tangents, (CO and AO), is measured by a semi-difference of arcs, concluded between its sides: \( \frac{\text{ABC} - \text{CDA}}{2} \). Products of segments of chords (AB and CD, Fig.51 or Fig.52), into which they are divided by an intersection point, are equal: \( \text{AO} \cdot \text{BO} = \text{CO} \cdot \text{DO} \).

![Fig. 50](image)

A square of tangent line segment is equal to a product of a secant line segment by the secant line external part (Fig.50): \( \text{OA}^2 = \text{OB} \cdot \text{OD} \) (prove, please!). This property may be considered as a particular case of Fig.52.

![Fig. 52](image)

A chord (AB, Fig.53), which is perpendicular to a diameter (CD), is divided into two in the intersection point O: \( \text{AO} = \text{OB} \). (Try to prove this!).
Inscribed and circumscribed polygons. Regular polygons

Regular polygon. Center of a regular polygon. Apothem.
Relations between sides and radii of a regular polygon.

Inscribed polygon in a circle is a polygon, vertices of which are placed on a circumference (Fig.54). Polygon circumscribed around a circle is a polygon, sides of which are tangents to the circumference (Fig.55).

Correspondingly, a circumference, going through vertices of a polygon (Fig.54), is called a circumcircle around a polygon; a circumference, for which sides of a polygon are tangents (Fig.55), is called an incircle into a polygon. For an arbitrary polygon it is impossible to inscribe a circle in it and to circumscribe a circle around it. For a triangle it is always possible. A radius $r$ of an incircle is expressed by sides $a$, $b$, $c$ of a triangle as:

$$ r = \sqrt{\frac{(p-a)(p-b)(p-c)}{p}} , \quad p = \frac{(a+b+c)}{2} . $$

A radius $R$ of a circumcircle is expressed by the formula:
It is possible to inscribe a circle in a quadrangle, if sums of its opposite sides are the same. In case of parallelograms it is valid only for a rhombus (a square). A center of an inscribed circle is placed in a point of intersection of diagonals. It is possible to circumscribe a circle around a quadrangle, if a sum of its opposite angles is equal to 180 deg. In case of parallelograms it is valid only for a rectangular (a square). A center of a circumscribed circle is placed in a point of intersection of diagonals. It is possible to circumscribe a circle around a trapezoid, only if it is an isosceles one.

Regular polygon is a polygon with equal sides and angles

On Fig. 56 a regular hexagon is shown, on Fig.57 – a regular octagon. A regular quadrangle is a square; a regular triangle is an equilateral triangle. Each angle of a regular polygon is equal to 180 \((n - 2) / n\) deg, where \(n\) is a number of angles. There is a point O (Fig. 56) inside of a regular polygon, equally removed from all its vertices (OA = OB = OC = … = OF), which is called a center of a regular polygon. The center is also equally removed from all the sides of a regular polygon (OP = OQ = OR = …). The segments OP, OQ, OR, … are called apothems; the segments OA, OB, OC, … – radii of a regular polygon. It is possible to inscribe a circle in a regular polygon and to circumscribe a circle around it. The centers of inscribed and circumscribed circles coincide with a center of a regular polygon. A radius of a circumscribed circle is a radius of a regular polygon, a radius of an inscribed circle is its apothem. The following formulas are relations between sides and radii of regular polygon:

\[
\begin{align*}
\text{for a regular triangle:} & \quad a = R \sqrt{3} \\
\text{for a regular quadrangle (a square):} & \quad a = R \sqrt{2} \\
\text{for a regular hexagon:} & \quad a = R.
\end{align*}
\]

For the most of regular polygons it is impossible to express the relation between their sides and radii by an algebraic formula.

Example. Is it possible to cut out a square with a side 30 cm from a circle with a diameter 40 cm?
Solution. The biggest square, included in a circle, is an inscribed square. According to the above mentioned formula its side is equal:

$$20 \sqrt{2} \approx 20 \cdot 1.41 \approx 28 \text{ cm}.$$ 

Hence, it is impossible to cut out a square with a side 30 cm from a circle with a diameter 40 cm.